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Complete description of polarization effects in e^+e^- pair production by a photon in the field of a strong laser wave

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Abstract. We consider production of a e^+e^- pair by a high-energy photon in the field of a strong laser wave. A probability of this process for circularly or linearly polarized laser photons and for arbitrary polarization of all other particles is calculated. We obtain the complete set of functions which describe such a probability in a compact invariant form. Besides, we discuss in some detail the polarization effects in the kinematics relevant to the problem of $e \to \gamma$ conversion at $\gamma\gamma$ and γe colliders.

1 Introduction

The e^+e^- pair production by two photons,

$$\gamma(k_1) + \gamma(k_2) \to e^+(p_+) + e^-(p_-)\,, \tag{1}$$

was calculated by Breit and Wheeler in 1934 (see, for example, the textbook [1], § 88). The polarization properties of this process were considered in [2–4]. With the growth of the laser field intensity, a high-energy photon starts to interact coherently with n laser photons,

$$\gamma(k_1) + n\,\gamma_{\rm L}(k_2) \to e^+(q_+) + e^-(q_-)\,, \tag{2}$$

thus the Breit–Wheeler process becomes non-linear. Such a process with absorption of $n = 4\div 5$ linearly polarized laser photons was observed in the recent experiment at SLAC [5]. The polarization properties of the process (2) are especially important for future $\gamma\gamma$ and γe colliders where this process can be a significant background for the laser conversion of $e \to \gamma$ in the conversion region (see [6,7] and the literature therein). In this case the non-linear Breit–Wheeler process must be taken into account in simulations of the processes in a conversion region. For comprehensive simulation, including processes of pair production and multiple electron scattering, one has to know not only the differential cross section of the process (2) with a given number of the absorbed laser photons n, but energy, angles and polarization of final positrons and electrons as well. Some particular polarization effects for this process were considered in [8–13] and have already been included in the existing simulation codes [12, 14].

In the present paper we give the complete description of the non-linear Breit–Wheeler process for the case of circularly or linearly polarized laser photons and arbitrary polarization of all other particles. For this purpose we exploit intensively the results of our recent paper [15] devoted to the cross-channel process – the non-linear Compton scattering[1]

$$e(q) + n\,\gamma_{\rm L}(k) \to e(q') + \gamma(k')\,. \tag{3}$$

Let e (e'^*) be the polarization 4-vector of the initial (final) photon and $u_{\mathbf p}$ ($\bar u_{\mathbf p'}$) be the bispinor of the initial (final) electron in the scattering amplitude of the process (3) while e_1 (e_2) be the polarization 4-vector of the initial high-energy (laser) photon and $v_{\mathbf p_+}$ ($\bar u_{\mathbf p_-}$) be the bispinor of the final positron (electron) in the scattering amplitude of the process (2). Then, basically, the results for the discussed process can be obtained from the corresponding results for the non-linear Compton scattering [15] by the replacements

$$\begin{aligned} &q \to -q_+\,,\ \ k \to k_2\,,\ \ q' \to q_-\,,\ \ k' \to -k_1\,,\\ &p \to -p_+\,,\ \ p' \to p_-\,, \end{aligned} \tag{4}$$

for the 4-momenta of particles involved and

$$\begin{aligned} &e(k) \to e_2(k_2)\,,\ \ e'^*(k') \to e_1(k_1)\,,\\ &u_{\mathbf p} \to v_{\mathbf p_+}\,,\ \ \bar u_{\mathbf p'} \to \bar u_{\mathbf p_-} \end{aligned} \tag{5}$$

for the amplitudes of their wave functions.

In the next section we describe in detail the kinematics of the process (2). The effective differential cross section is obtained in Sect. 3 in the compact invariant form, including the polarization of all particles. In Sect. 4 we consider the process (2) in the reference frame relevant for the conversion

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[1] Below we shall quote formulae from this paper by a double numbering, for example, (1.21) means (21) from [15].

region at the $\gamma\gamma$ and γe colliders. The limiting cases are discussed in Sect. 5. In Sect. 6 we present some numerical results obtained for the range of parameters close to those in the existing TESLA project [16]. In the last section we summarize our results and compare them with those known in the literature. In the appendix we consider the limit of a weak laser field and present in a compact form the cross section of the process (1) with all four particles polarized.

We use the system of units in which the velocity of light $c = 1$ and the Plank constant $\hbar = 1$. In what follows, we will often consider the non-linear Breit–Wheeler process in the frame of reference in which a high-energy photon performs a head-on collision with laser photons, i.e. in which $\mathbf{k}_1 \parallel (-\mathbf{k}_2)$. We call this the "collider system". In this frame of reference we choose the z-axis along the high-energy photon momentum $\mathbf{k}_1$. All azimuthal angles are defined with respect to one fixed x-axis: $\varphi_\pm$ of the final leptons, $\beta_\pm$ of its polarization vectors $\boldsymbol{\zeta}_\pm$ and γ_1 (γ_2) of the direction of the high-energy photon (laser photon) linear polarization.

2 Kinematics

2.1 Invariant variables

The invariant parameter describing the intensity of the laser field (the parameter of non-linearity) is defined via the mean value of squared 4-potential $A_\mu(x)$:

$$\xi^2 = -\frac{e^2}{m^2}\,\langle A_\mu(x)A^\mu(x)\rangle\,, \tag{6}$$

where e and m are the electron charge and the mass. We use this definition of ξ^2 both for the circularly and linearly polarized laser photons. Other useful expressions for this parameter are given by (1.4)–(1.5). When describing the non-linear process (2), one has to take into account that in a laser wave 4-momenta $p_\pm$ of the free $e^\pm$ are replaced by the 4-quasi-momenta $q_\pm$:

$$q_\pm = p_\pm + \xi^2\frac{m^2}{2k_2p_\pm}\,k_2\,, \tag{7}$$
$$q_\pm^2 = (1+\xi^2)\,m^2 \equiv m_*^2\,.$$

In particular, the energy of the free positron or electron $E_\pm$ is replaced by the quasi-energy

$$(q_\pm)_0 = E_\pm + \xi^2\frac{m^2}{2k_2p_\pm}\,\omega_2\,. \tag{8}$$

As a result, we deal with the reaction (2) for which the conservation law reads

$$k_1 + n\,k_2 = q_+ + q_-\,. \tag{9}$$

Note that $k_2q_\pm = k_2p_\pm$. It is convenient to use the following invariant variables:

$$x = \frac{2k_1k_2}{m^2}\,,\quad y_\pm = \frac{k_2p_\pm}{k_1k_2}\,,\quad y_+ + y_- = 1\,. \tag{10}$$

The threshold x_n of the process (2) is determined from the equality

$$(k_1+n\,k_2)^2 = m^2\,x_n\,n = (2m_*)^2\,,\quad x_n = \frac{4(1+\xi^2)}{n}\,; \tag{11}$$

therefore, when $x < x_1 = 4(1+\xi^2)$, the process is possible only if the number of the absorbed laser photon n is larger then the threshold value

$$n_{\rm th} = \frac{4(1+\xi^2)}{x}\,. \tag{12}$$

Further, we introduce the auxiliary combinations

$$s_n = 2\sqrt{r_n(1-r_n)},\quad c_n = 1-2r_n\,,\quad s_n^2+c_n^2 = 1\,, \tag{13}$$

where

$$r_n = \frac{1+\xi^2}{nx\,y_+y_-}\,. \tag{14}$$

It is useful to note that these invariants have a simple notion in the collider frame of reference, namely

$$r_n = \frac{m_*^2}{m_\perp^2}\,,\quad s_n = \frac{2m_*|\mathbf{p}_\perp|}{m_\perp^2}\,,\quad c_n = \frac{\mathbf{p}_\perp^2 - m_*^2}{m_\perp^2}\,, \tag{15}$$

where

$$m_\perp^2 = m_*^2 + \mathbf{p}_\perp^2 = 2\,\frac{(k_1q_+)\,(k_1q_-)}{n\,(k_1k_2)} \tag{16}$$

and $\mathbf{p}_\perp$ is the transverse momentum of the positron in this system:

$$\mathbf{p}_\perp \equiv (\mathbf{p}_+)_\perp = -\,(\mathbf{p}_-)_\perp\,. \tag{17}$$

Therefore,

$$0 \le s_n < 1\,;\quad 0 < r_n \le 1\,;\quad -1 \le c_n < 1\,. \tag{18}$$

The minimum and maximum values of the variable $y_\pm$ for the reaction (2) are

$$\max\{y_\pm\} = y_n = \frac{1}{2}\,(1+v_n)\,,\quad v_n = \sqrt{1-\frac{4(1+\xi^2)}{nx}}\,,$$
$$\min\{y_\pm\} = 1-y_n = \frac{1}{2}\,(1-v_n) \tag{19}$$

(here v_n is the quasi-velocity of $e^\pm$ in the center-of-mass system). The value of y_n is close to 1 for large n, but for a given n it decreases with the growth of the non-linearity parameter ξ^2. With this notation one can rewrite r_n and s_n in the form

$$r_n = \frac{y_n(1-y_n)}{y_+y_-}\,,$$
$$s_n = 2\sqrt{y_n(1-y_n)}\;\frac{\sqrt{(y_n-y_+)(y_n-y_-)}}{y_+y_-}\,, \tag{20}$$

from which it follows that

$$s_n \to 0 \quad\text{at}\quad y_\pm \to y_n \;\text{ or at }\; y_\pm \to 1-y_n\,. \tag{21}$$

It is useful to note that the invariants $x,\ y$ defined in (1.9) are connected with the corresponding invariants introduced in the present paper (10) by the replacements (4) under which we have

$$x \to -xy_+\,, \quad y \to \frac{1}{y_+}\,. \tag{22}$$

Besides, the combinations of these invariants $s_n,\ c_n,\ r_n$ defined in (1.10) and (1.11) are replaced by the corresponding combinations (13) and (14) by the rules

$$s_n \to s_n\,, \quad c_n \to c_n\,, \quad r_n \to r_n\,. \tag{23}$$

2.2 Invariant polarization parameters

The invariant description of the polarization properties of both the high-energy and laser photons can be performed in the same way as in [15]. We define a pair of unit 4-vectors[2]

$$e^{(1)} = \frac{N}{-\sqrt{-N^2}}\,, \quad e^{(2)} = \frac{P}{\sqrt{-P^2}}\,, \tag{24}$$

where

$$\begin{aligned}
N^\mu &= \varepsilon^{\mu\alpha\beta\gamma} P_\alpha(-k_1 - n\,k_2)_\beta K_\gamma\,, \quad \varepsilon^{0123} = +1\,,\\
P_\alpha &= (q_- - q_+)_\alpha - \frac{(q_- - q_+)K}{K^2}\,K_\alpha\,,\\
K_\alpha &= (n\,k_2 - k_1)_\alpha\,,\\
\sqrt{-N^2} &= m^2 x\,n\,\sqrt{-P^2}\,, \quad \sqrt{-P^2} = m\,\frac{s_n}{r_n}\sqrt{1+\xi^2}\,.
\end{aligned}$$

The 4-vectors $e^{(1)}$ and $e^{(2)}$ are orthogonal to each other and to the 4-vectors k_1 and k_2:

$$e^{(i)}e^{(j)} = -\delta_{ij}\,, \quad e^{(i)}k_1 = e^{(i)}k_2 = 0; \quad i,\, j = 1,\, 2\,. \tag{25}$$

In the collider system they have only a transverse component:

$$\begin{aligned}
\mathbf{e}_\perp^{(1)} &= \frac{\mathbf{p}_\perp \times \mathbf{k}_1}{|\mathbf{p}_\perp \times \mathbf{k}_1|}\,, \quad \mathbf{e}_\perp^{(2)} = -\frac{\mathbf{p}_\perp}{|\mathbf{p}_\perp|}\,,\\
e_0^{(j)} &= e_z^{(j)} = 0\,.
\end{aligned} \tag{26}$$

Therefore, the vector $\mathbf{e}^{(2)}$ is in the plane of $\mathbf{k}_1$ and $\mathbf{p}_+$ (the plane of scattering) and $\mathbf{e}^{(1)}$ is perpendicular to that plane. Note that the vectors $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$ and $\mathbf{k}_2$ form a right-handed set as well as the vectors $\mathbf{e}^{(1)}$, $(-\mathbf{e}^{(2)})$ and $\mathbf{k}_1$.

Let ξ_j be the Stokes parameters for the high-energy photon which are defined with respect to the 4-vectors $e^{(1)}$ and $(-e^{(2)})$ and $\tilde\xi_j$ be the Stokes parameters for the laser photon which are defined with respect to the 4-vectors $e^{(1)}$ and $e^{(2)}$.

[2] In this definition we take into account that at the replacements (4) we have for vectors (1.19) the following replacements: $N_\mu \to N_\mu$, $P_\mu \to P_\mu$, $\sqrt{-P^2} \to \sqrt{-P^2}$, but $\sqrt{-N^2} = m^2nxy\sqrt{-P^2} \to -m^2nx\sqrt{-P^2} = -\sqrt{-N^2}$.

These parameters are related to the Stokes parameters which were used in the description of the non-linear Compton scattering [15]. At the replacements (5) the Stokes parameters ξ_j and ξ'_j for the laser and final photon in the non-linear Compton scattering are replaced by the Stokes parameters for the process (2) according to the rules:

$$\xi_j \to \tilde\xi_j\,, \quad \xi'_{1,\,3} \to \xi_{1,\,3}\,, \quad \xi'_2 \to -\xi_2\,. \tag{27}$$

The minus sign in the last rule, $\xi'_2 \to -\xi_2$, is due to the fact that the polarization matrix of the final photon in the Compton scattering $\rho'^{\mu\nu}$ is replaced at the replacements (5) by the matrix of the initial high-energy photon $\rho_1^{\nu\mu}$.

As for the polarization of the final positron and electron, it is necessary to distinguish the polarization vector $\boldsymbol{\zeta}_\pm^{(f)}$ of the final $e^\pm$ as resulting from the scattering process itself from the detected polarization $\boldsymbol{\zeta}_\pm$ which enters the effective cross section and which essentially represents the properties of the detector as selecting one or other polarization of the final lepton (for details see [1], § 65). The vector $\boldsymbol{\zeta}_\pm$ also determines the lepton-spin 4-vector

$$a_\pm = \left(\frac{\boldsymbol{\zeta}_\pm \mathbf{p}_\pm}{m}\,,\, \boldsymbol{\zeta}_\pm + \frac{\mathbf{p}_\pm\,(\boldsymbol{\zeta}_\pm \mathbf{p}_\pm)}{m(E_\pm + m)}\right) \tag{28}$$

and the mean helicity of the final leptons

$$\langle\lambda_\pm\rangle = \frac{\boldsymbol{\zeta}_\pm \mathbf{p}_\pm}{2|\mathbf{p}_\pm|}\,. \tag{29}$$

Now we have to define invariants which describe the polarization properties of the final positrons and electrons. Similar to the approach in [15], we define the two sets of units 4-vectors:

$$\begin{aligned}
e_1^\pm &= \mp\, e^{(1)}\,, \quad e_2^\pm = \pm\, e^{(2)} \mp \frac{\sqrt{-P^2}}{m^2xy_\pm}\,k_2\,,\\
e_3^\pm &= \frac{1}{m}\left(p_\pm - \frac{2}{xy_\pm}\,k_2\right)\,.
\end{aligned} \tag{30}$$

These vectors satisfy the conditions

$$\begin{aligned}
e_i^+e_j^+ &= -\delta_{ij}\,, \quad e_j^+p_+ = 0\\
e_i^-e_j^- &= -\delta_{ij}\,, \quad e_j^-p_- = 0\,,
\end{aligned} \tag{31}$$

besides $e_j^- \leftrightarrow e_j^+$ under the exchange electron $\leftrightarrow$ positron.

It allows us to represent the 4-vectors $a_\pm$ in the following covariant form:

$$a_\pm = \sum_{j=1}^{3} \zeta_j^\pm e_j^\pm\,, \tag{32}$$

where

$$\zeta_j^\pm = -a_\pm e_j^\pm\,. \tag{33}$$

The invariants $\zeta_j^\pm$ describe completely the detected polarization properties of the final $e^\pm$. It is not difficult to find that $\zeta_1^\pm$ is the polarization perpendicular to the scattering plane. Besides, in the frame of reference, in which the lepton momentum $\mathbf{p}_\pm$ is anti-parallel to the laser photon momentum $\mathbf{k}_2$, the invariant $\zeta_3^\pm = 2\langle\lambda_\pm\rangle$. Furthermore, we will show that for the practically important case, relevant for the $e \to \gamma$ conversion, this frame of reference almost coincides with the collider system.

Note that the invariants $\zeta_j^\pm$ are connected with the invariants ζ_j and ζ_j' for the initial and final electron in the non-linear Compton scattering. The corresponding rules are

$$\zeta_j \to -\zeta_j^+ \,, \;\; \zeta_j' \to \zeta_j^- \,. \tag{34}$$

The minus sign in the first rule, $\zeta_j \to -\zeta_j^+$, is due to the fact that at the replacements (5) the unit 4-vectors for the initial electron in the Compton scattering e_j is replaced by the unit 4-vectors of the final positron e_j^+ with an additional minus sign, $e_j \to -e_j^+$.

To clarify the meaning of invariants $\zeta_j^\pm$, it is useful to note that

$$\zeta_j^\pm = \boldsymbol{\zeta}_\pm \, \mathbf{n}_j^\pm \,, \tag{35}$$

where the corresponding 3-vectors are

$$\mathbf{n}_j^\pm = \mathbf{e}_j^\pm - \frac{\mathbf{p}_\pm}{E_\pm + m} \, e_{j0}^\pm \tag{36}$$

with $e_{j0}^\pm$ being a time component of the 4-vector $e_j^\pm$ defined in (30). Using the properties (31) of the 4-vectors $e_j^\pm$, one can check that

$$\mathbf{n}_i^+ \, \mathbf{n}_j^+ = \delta_{ij} \,, \;\; \mathbf{n}_i^- \, \mathbf{n}_j^- = \delta_{ij} \,. \tag{37}$$

As a result, the polarization vector $\boldsymbol{\zeta}_\pm$ has the form

$$\boldsymbol{\zeta}_\pm = \sum_{j=1}^{3} \zeta_j^\pm \, \mathbf{n}_j^\pm \,. \tag{38}$$

3 Cross section in the invariant form

3.1 General relations

The usual notion of the cross section is not applicable for the reaction (2) and usually its description is given in terms of the probability of the process per second $\dot{W}^{(n)}$. However, for the procedure of simulation in the conversion region as well as for the simple comparison with the linear process (1), it is useful to introduce the "effective cross section" given by the definition

$$\mathrm{d}\sigma^{(n)} = \frac{\mathrm{d}\dot{W}^{(n)}}{j} \,, \tag{39}$$

where

$$j = \frac{(k_1\, k_2)}{\omega_1\, \omega_2}\, n_{\mathrm{L}} = \frac{m^2\, x}{2\omega_1 \omega_2}\, n_{\mathrm{L}} \tag{40}$$

is the flux density of colliding particles (and n_{L} is the density of photons in the laser wave). Contrary to the usual cross section, this effective cross section does depend on the laser beam intensity, i.e. on the parameter ξ^2. The total effective cross section is defined as the sum over harmonics, corresponding to the reaction (2) with a given number n of absorbed laser photons[3]:

$$\mathrm{d}\sigma = \sum_n \mathrm{d}\sigma^{(n)} \,. \tag{41}$$

The effective differential cross section with arbitrary polarization of all particles can be presented in the following invariant form:

$$\mathrm{d}\sigma(\boldsymbol{\xi},\, \tilde{\boldsymbol{\xi}},\, \boldsymbol{\zeta}_\pm) = \frac{r_e^2}{4x} \, \sum_n \bar{F}^{(n)} \, \mathrm{d}\varGamma_n \,, \tag{42}$$

$$\mathrm{d}\varGamma_n = \delta(k_1 + n\, k_2 - q_+ - q_-) \, \frac{\mathrm{d}^3 q_+}{(q_+)_0} \, \frac{\mathrm{d}^3 q_-}{(q_-)_0} \,,$$

where $r_e = \alpha/m$ is the classical electron radius, and $\bar{F}$ can be obtained from the corresponding function F for the non-linear Compton scattering (1.32) by relation[4]

$$\bar{F}^{(n)} = -F^{(n)} \,, \tag{43}$$

and in the function $F^{(n)}$, defined by (1.33), (1.46)–(1.52) and (1.71)–(1.76), the following replacements, discussed in the previous section, have to be done:

$$\begin{aligned} & x \to -x y_+ \,, \;\; y \to \frac{1}{y_+} \,, \\ & r_n \to r_n \,, \;\; s_n \to s_n \,, \;\; c_n \to c_n \,, \\ & \xi_j \to \tilde{\xi}_j \,, \; \xi_{1,3}' \to \xi_{1,3} \,, \; \xi_2' \to -\xi_2 \,, \; \zeta_j \to -\zeta_j^+ \,, \\ & \zeta_j' \to \zeta_j^- \,. \end{aligned} \tag{44}$$

We present the function $\bar{F}^{(n)}$ in the form

$$\bar{F}^{(n)} = \bar{F}_0^{(n)} + \sum_{j=1}^{3} \Big(G_j^{(n)+} \zeta_j^+ \; + \; G_j^{(n)-} \zeta_j^- \Big) + \sum_{i,j=1}^{3} \bar{H}_{ij}^{(n)} \, \zeta_i^+ \, \zeta_j^- \,. \tag{45}$$

Here the function $\bar{F}_0^{(n)}$ describes the total cross section for a given harmonic n, summed over spin states of the final particles:

$$\sigma^{(n)}(\boldsymbol{\xi},\, \tilde{\boldsymbol{\xi}}) = \frac{r_e^2}{x} \int \bar{F}_0^{(n)} \, \mathrm{d}\varGamma_n \,. \tag{46}$$

The terms $G_j^{(n)+} \zeta_j^+$ and $G_j^{(n)-} \zeta_j^-$ in (45) describe the polarization of the final positrons and the final electrons,

[3] In this formula and below the sum is over those n which satisfy the condition $y < y_n$, i.e. this sum runs from some minimal value $n_{\min}$ up to $n = \infty$, where $n_{\min}$ is determined by the equation $y_{n_{\min}-1} < y < y_{n_{\min}}$.

[4] The sign minus in this equation is due to the fact that the polarization matrix of the initial electron in the Compton scattering $(\hat{p} + m)\,(1 - \gamma^5 \hat{a})/2$ is replaced at the transition (5) to the matrix $(-\hat{p}_+ + m)\,(1 - \gamma^5 \hat{a}_+)/2$, which is the polarization matrix of the final positron times (-1).

respectively. The last terms $\bar{H}_{ij}^{(n)}\zeta_i^+\zeta_j^-$ stand for the correlation of the final particles' polarizations.

From (45) one can deduce the polarization of the final positron $\zeta_j^{(f)+}$ and electron $\zeta_j^{(f)-}$ resulting from the scattering process itself. According to the usual rules (see [1], § 65), we obtain the following expression for the polarization of the final positron (electron) (summed over polarization states of the final electron (positron)):

$$\zeta_j^{(f)\pm} = \frac{G_j^\pm}{\bar{F}_0}\,, \quad \bar{F}_0 = \sum_n \bar{F}_0^{(n)}\,, \quad G_j^\pm = \sum_n G_j^{(n)\pm}\,,$$
$$j = 1,\, 2,\, 3\,; \tag{47}$$

therefore, its polarization vector is

$$\boldsymbol{\zeta}_\pm^{(f)} = \sum_{j=1}^{3} \frac{G_j^\pm}{\bar{F}_0}\,\mathbf{n}_j^\pm\,. \tag{48}$$

In the similar way, the polarization properties for a given harmonic n are described by

$$\zeta_j^{(n)(f)\pm} = \frac{G_j^{(n)\pm}}{\bar{F}_0^{(n)}}\,. \tag{49}$$

3.2 The results for the circularly polarized laser photons

In this subsection we consider the case of 100% circularly polarized laser beam with the Stokes parameters

$$\tilde{\xi}_1 = \tilde{\xi}_3 = 0, \quad \tilde{\xi}_2 = P_c = \pm 1\,. \tag{50}$$

In the considered case almost all dependence on the nonlinearity parameter ξ^2 accumulates in three functions:

$$\begin{aligned} f_n &\equiv f_n(z_n) = J_{n-1}^2(z_n) + J_{n+1}^2(z_n) - 2J_n^2(z_n)\,, \\ g_n &\equiv g_n(z_n) = \frac{4n^2 J_n^2(z_n)}{z_n^2}\,, \\ h_n &\equiv h_n(z_n) = J_{n-1}^2(z_n) - J_{n+1}^2(z_n)\,, \end{aligned} \tag{51}$$

where $J_n(z)$ is the Bessel function. The functions (51) depend on x, $y_\pm$ and ξ^2 via the single argument

$$z_n = \frac{\xi}{\sqrt{1+\xi^2}}\, n\, s_n\,. \tag{52}$$

For the small value of this argument one has

$$f_n = g_n = h_n = \frac{(z_n/2)^{2(n-1)}}{[(n-1)!]^2} \quad \text{at} \;\; z_n \to 0\,, \tag{53}$$

in particular,

$$f_1 = g_1 = h_1 = 1 \quad \text{at} \;\; z_1 = 0\,. \tag{54}$$

It is useful to note that this argument is small for small ξ^2, as well as for the minimum or maximum values of $y_\pm$:

$$z_n \to 0 \;\; \text{either at} \;\; \xi^2 \to 0\,, \tag{55}$$
$$\text{or at} \quad y_\pm \to 1 - y_n\,, \quad \text{or at} \quad y_\pm \to y_n\,.$$

The results of our calculations are the following. The function $\bar{F}_0^{(n)}$, related to the total cross section (46), reads

$$\begin{aligned} \bar{F}_0^{(n)} &= (u-2)\, f_n + \frac{s_n^2}{1+\xi^2}\, g_n - (u-2)\, c_n h_n P_c\, \xi_2 \\ &\quad - \left[2(f_n - g_n) + s_n^2(1+\varDelta)\, g_n\right] \xi_3\,. \end{aligned} \tag{56}$$

Here and below we use the notation

$$u = \frac{1}{y_+\, y_-}\,, \quad \varDelta = \frac{\xi^2}{1+\xi^2}\,. \tag{57}$$

The polarization of the final positrons $\zeta_j^{(f)+}$ is given by (47)– (49) with

$$\begin{aligned} G_1^{(n)+} &= -\frac{1}{y_-}\frac{s_n}{\sqrt{1+\xi^2}}\, h_n P_c\, \xi_1\,, \\ G_2^{(n)+} &= \frac{s_n}{y_+\sqrt{1+\xi^2}}\,(c_n g_n \xi_2 - h_n\, P_c) \\ &\quad + \frac{1}{y_-}\frac{s_n}{\sqrt{1+\xi^2}}\, h_n P_c\, \xi_3\,, \\ G_3^{(n)+} &= -(y_+ - y_-) u\, c_n\, h_n\, P_c \\ &\quad + u\left[(y_+ - y_-)\, f_n + \frac{s_n^2\, y_-}{1+\xi^2}\, g_n\right] \xi_2\,. \end{aligned} \tag{58}$$

The polarization of the final electrons $\zeta_j^{(f)-}$ is given by (47)– (49) with

$$\begin{aligned} G_1^{(n)-} &= -\frac{s_n}{y_+\sqrt{1+\xi^2}} h_n P_c\, \xi_1\,, \\ G_2^{(n)-} &= \frac{s_n}{y_-\sqrt{1+\xi^2}}\,(c_n g_n \xi_2 - h_n\, P_c) \\ &\quad + \frac{s_n}{y_+\sqrt{1+\xi^2}}\, h_n P_c\, \xi_3\,, \\ G_3^{(n)-} &= (y_+ - y_-) u c_n\, h_n P_c \\ &\quad - u\left[(y_+ - y_-) f_n - \frac{s_n^2\, y_+}{1+\xi^2} g_n\right] \xi_2\,. \end{aligned} \tag{59}$$

Correlations of the electron and positron polarizations are given by the functions

$$\begin{aligned} \bar{H}_{11}^{(n)} &= 2f_n - \frac{s_n^2}{1+\xi^2}\, g_n - 2c_n h_n P_c\, \xi_2 \\ &\quad - \{(u-2)(f_n - g_n) \\ &\qquad - [1 - (u-1)\varDelta]\, s_n^2 g_n\}\, \xi_3\,, \\ \bar{H}_{12}^{(n)} &= u\,[(y_+ - y_-)(f_n - g_n) \\ &\qquad + (y_+ - y_-\varDelta)\, s_n^2\, g_n]\ \xi_1\,, \\ \bar{H}_{13}^{(n)} &= -\frac{c_n s_n}{y_-\sqrt{1+\xi^2}} g_n\, \xi_1\,, \end{aligned}$$

$$\begin{aligned}
\bar{H}_{21}^{(n)} &= -u\,\left[(y_+ - y_-)\,(f_n - g_n) - (y_- - y_+\Delta)\,s_n^2\,g_n\right]\xi_1\,,\\
\bar{H}_{22}^{(n)} &= 2f_n - \frac{s_n^2}{1+\xi^2}\,g_n - 2c_n h_n P_c\,\xi_2 - \left[(u-2)(f_n - g_n) + (u-1-\Delta)\,s_n^2\,g_n\right]\xi_3\,,\\
\bar{H}_{23}^{(n)} &= \frac{s_n}{y_+\sqrt{1+\xi^2}}\,(c_n g_n - h_n P_c\,\xi_2) + \frac{c_n s_n}{y_-\sqrt{1+\xi^2}}\,g_n\,\xi_3\,,\\
\bar{H}_{31}^{(n)} &= -\frac{c_n s_n}{y_+\sqrt{1+\xi^2}}\,g_n\,\xi_1\,,\\
\bar{H}_{32}^{(n)} &= \frac{s_n}{y_-\sqrt{1+\xi^2}}\,(c_n\,g_n - h_n P_c\,\xi_2) + \frac{c_n s_n}{y_+\sqrt{1+\xi^2}}\,g_n\,\xi_3\,,\\
\bar{H}_{33}^{(n)} &= -(u-2)\,f_n + (u-1)\,\frac{s_n^2}{1+\xi^2}\,g_n + (u-2)\,c_n h_n P_c\,\xi_2 + \left[2(f_n - g_n) + (1+\Delta)\,s_n^2\,g_n\right]\xi_3\,.
\end{aligned} \tag{60}$$

It should be noted that among the presented functions there are defined relations connected with the symmetry under the exchange electron $\leftrightarrow$ positron, namely, under the replacement

$$y_+ \leftrightarrow y_- \tag{61}$$

one has

$$G_j^{(n)+} \leftrightarrow G_j^{(n)-}\,,\quad \bar{H}_{ij}^{(n)} \leftrightarrow \bar{H}_{ji}^{(n)}\,. \tag{62}$$

3.3 The results for the linearly polarized laser photons

Here we consider the case of 100% linearly polarized laser beam. In this case, the electromagnetic laser field is described by the 4-potential

$$A^\mu(x) = A^\mu\,\cos(kx)\,,\quad A^\mu = \frac{\sqrt{2}\,m}{e}\,\xi\,e_{\mathrm{L}}^\mu\,,\qquad e_{\mathrm{L}}e_{\mathrm{L}} = -1\,, \tag{63}$$

where e_{L}^μ is the unit 4-vector describing the polarization of the laser photons, which can be expressed in the covariant form via the unit 4-vectors $e^{(1,2)}$ given in (24) as follows:

$$e_{\mathrm{L}} = e^{(1)}\,\sin\varphi - e^{(2)}\,\cos\varphi\,. \tag{64}$$

The Stokes parameters $\tilde{\xi}_i$ of the laser photon are defined with respect to the 4-vectors $e^{(1)}$ and $e^{(2)}$ and are equal to

$$\tilde{\xi}_1 = -\sin 2\varphi,\quad \tilde{\xi}_2 = 0,\quad \tilde{\xi}_3 = -\cos 2\varphi\,. \tag{65}$$

The invariants

$$\sin\varphi = -e_{\mathrm{L}}\,e^{(1)}\,,\quad \cos\varphi = e_{\mathrm{L}}\,e^{(2)} \tag{66}$$

have a simple notion in the collider system in which the vectors $e^{(1,2)}$ are given by (26). For the problem discussed it is convenient to choose the x-axis of this frame of reference along the direction of the laser linear polarization, i.e. along the vector $\mathbf{e}_{\mathrm{L}}$. With such a choice, the quantity φ is the azimuthal angle of the final positron φ_+ in the collider system (analogously, the azimuthal angle $\beta_\pm$ of the $e^\pm$ polarization $\boldsymbol{\zeta}_\pm$ is also defined with respect to this x-axis).

When calculating the effective cross section, we found that almost all dependence on the non-linearity parameter ξ^2 accumulates in three functions:

$$\begin{aligned}
\tilde{f}_n &= 4\left[A_1(n,\,a,\,b)\right]^2 - 4A_0(n,\,a,\,b)A_2(n,\,a,\,b)\,,\\
\tilde{g}_n &= \frac{4n^2}{z_n^2}\,\left[A_0(n,\,a,\,b)\right]^2\,,\\
\tilde{h}_n &= \frac{4n}{a}\,A_0(n,\,a,\,b)\,A_1(n,\,a,\,b)\,,
\end{aligned} \tag{67}$$

where

$$A_k(n,\,a,\,b) = \int_{-\pi}^{\pi} \cos^k\psi\,\exp\left[\mathrm{i}\,(n\psi - a\sin\psi + b\sin 2\psi)\right]\frac{\mathrm{d}\psi}{2\pi}\,. \tag{68}$$

The arguments of these functions are

$$a = -z_n\,\sqrt{2}\,\cos\varphi\,,\quad b = \frac{\xi^2}{2xy_+y_-}\,, \tag{69}$$

where z_n is defined in (52). To find the positron spectrum, one needs also the functions (67) averaged over the azimuthal angle φ:

$$\langle\tilde{f}_n\rangle = \int_0^{2\pi}\tilde{f}_n\,\frac{\mathrm{d}\varphi}{2\pi}\,,\quad \langle\tilde{g}_n\rangle = \int_0^{2\pi}\tilde{g}_n\,\frac{\mathrm{d}\varphi}{2\pi}\,. \tag{70}$$

For small values of $\xi^2 \to 0$ one has

$$\tilde{f}_n,\,\tilde{g}_n,\,\tilde{h}_n \propto \left(\xi^2\right)^{n-1}\,, \tag{71}$$

in particular, at $\xi^2 = 0$

$$\tilde{f}_1 = \langle\tilde{f}_1\rangle = \langle\tilde{g}_1\rangle = \tilde{h}_1 = 1\,,\quad \tilde{g}_1 = 1 + \cos 2\varphi\,. \tag{72}$$

The results of our calculations are the following. First, we define the auxiliary functions

$$\begin{aligned}
X_n &= \tilde{f}_n - (1+c_n)\left[(1-\Delta\,r_n)\,\tilde{g}_n - \tilde{h}_n\,\cos 2\varphi\right]\,,\\
Y_n &= (1+c_n)\tilde{g}_n - 2\tilde{h}_n\cos^2\varphi\,,\\
V_n &= \tilde{f}_n\,\cos 2\varphi + 2(1+c_n)\left[(1-\Delta\,r_n)\,\tilde{g}_n - 2\tilde{h}_n\,\cos^2\varphi\right]\sin^2\varphi\,,
\end{aligned} \tag{73}$$

where c_n, r_n and Δ are defined in (13), (14) and (57), respectively.

The function $\bar{F}_0^{(n)}$, related to the total cross section (46), reads

$$\bar{F}_0^{(n)} = (u-2)\,\tilde{f}_n + \frac{s_n^2}{1+\xi^2}\,\tilde{g}_n - 2\,X_n\,\xi_1 \sin 2\varphi + \left(2\,V_n - \frac{s_n^2}{1+\xi^2}\,\tilde{g}_n\right)\xi_3\,. \tag{74}$$

The polarization of the final positrons $\zeta_j^{(f)+}$ is given by (47)– (49) with

$$\begin{aligned} G_1^{(n)+} &= -\frac{s_n}{y_+\sqrt{1+\xi^2}}\,\tilde{h}_n\,\xi_2 \sin 2\varphi\,, \\ G_2^{(n)+} &= \frac{s_n}{y_+\sqrt{1+\xi^2}}\,Y_n\,\xi_2\,, \\ G_3^{(n)+} &= u\left[(y_+-y_-)\,\tilde{f}_n + \frac{s_n^2\,y_-}{1+\xi^2}\,\tilde{g}_n\right]\xi_2\,. \end{aligned} \tag{75}$$

The polarization of the final electrons $\zeta_j^{(f)-}$ is given by (47)– (49) with

$$\begin{aligned} G_1^{(n)-} &= -\frac{s_n}{y_-\sqrt{1+\xi^2}}\,\tilde{h}_n\,\xi_2 \sin 2\varphi\,, \\ G_2^{(n)-} &= \frac{s_n}{y_-\sqrt{1+\xi^2}}\,Y_n\,\xi_2\,, \\ G_3^{(n)-} &= -u\left[(y_+-y_-)\tilde{f}_n - \frac{s_n^2\,y_+}{1+\xi^2}\,\tilde{g}_n\right]\xi_2\,. \end{aligned} \tag{76}$$

Correlations of the electron and positron polarizations are given by the functions

$$\begin{aligned} \bar{H}_{11}^{(n)} &= \left[2\tilde{f}_n - \frac{s_n^2}{1+\xi^2}\,\tilde{g}_n\right] - (u-2)\,X_n\,\xi_1 \sin 2\varphi \\ &\quad + \left[(u-2)V_n + \frac{s_n^2}{1+\xi^2}\,\tilde{g}_n\right]\xi_3\,, \\ \bar{H}_{12}^{(n)} &= -u\left[(y_+-y_-)V_n - \frac{s_n^2\,y_+}{1+\xi^2}\,\tilde{g}_n\right]\xi_1 \\ &\quad - u(y_+-y_-)\,X_n\,\xi_3 \sin 2\varphi\,, \\ \bar{H}_{13}^{(n)} &= -\frac{s_n}{y_+\sqrt{1+\xi^2}}\,\tilde{h}_n \sin 2\varphi \\ &\quad - \frac{s_n}{y_-\sqrt{1+\xi^2}}\left(Y_n\,\xi_1 - \tilde{h}_n\,\xi_3 \sin 2\varphi\right), \\ \bar{H}_{21}^{(n)} &= u\left[(y_+-y_-)\,V_n + \frac{s_n^2\,y_-}{(1+\xi^2)}\,\tilde{g}_n\right]\xi_1 \\ &\quad + u(y_+-y_-)\,X_n\,\xi_3 \sin 2\varphi\,, \\ \bar{H}_{22}^{(n)} &= \left[2\tilde{f}_n - \frac{s_n^2}{1+\xi^2}\,\tilde{g}_n\right] - (u-2)X_n\,\xi_1 \sin 2\varphi \\ &\quad + \left[(u-2)V_n - (u-1)\frac{s_n^2}{1+\xi^2}\,\tilde{g}_n\right]\xi_3\,, \\ \bar{H}_{23}^{(n)} &= \frac{s_n}{y_+\sqrt{1+\xi^2}}\,Y_n \\ &\quad + \frac{s_n}{y_-\sqrt{1+\xi^2}}\left(\tilde{h}_n\,\xi_1 \sin 2\varphi + Y_n\,\xi_3\right), \\ \bar{H}_{31}^{(n)} &= -\frac{s_n}{y_-\sqrt{1+\xi^2}}\,\tilde{h}_n \sin 2\varphi \\ &\quad - \frac{s_n}{y_+\sqrt{1+\xi^2}}\left(Y_n\,\xi_1 - \tilde{h}_n\,\xi_3 \sin 2\varphi\right), \\ \bar{H}_{32}^{(n)} &= \frac{s_n}{y_-\sqrt{1+\xi^2}}\,Y_n \\ &\quad + \frac{s_n}{y_+\sqrt{1+\xi^2}}\left(\tilde{h}_n\,\xi_1 \sin 2\varphi + Y_n\,\xi_3\right), \\ \bar{H}_{33}^{(n)} &= -\left[(u-2)\tilde{f}_n - (u-1)\frac{s_n^2}{1+\xi^2}\tilde{g}_n\right] \\ &\quad + 2X_n\xi_1 \sin 2\varphi - \left(2\,V_n - \frac{s_n^2}{1+\xi^2}\,\tilde{g}_n\right)\xi_3\,. \end{aligned} \tag{77}$$

The presented functions obey the same relations (62), connected with the symmetry (61) under the exchange electron $\leftrightarrow$ positron, as for the case of the circularly polarized laser photons.

4 Going to the collider system

4.1 Exact relations

As an example of the application of the above formulae, let us consider the non-linear Breit–Wheeler process in the collider system defined in Sect. 1. In such a frame of reference the invariants (10) and the phase volume element in (42) are equal to

$$x = \frac{4\omega_1\omega_2}{m^2}\,, \quad y_\pm = \frac{E_\pm + (p_\pm)_z}{2\omega_1}\,, \quad \mathrm{d}\varGamma_n = \mathrm{d}y_+\,\mathrm{d}\varphi_+\,, \tag{78}$$

and the differential cross section, summed over spin states of the final particles, is

$$\frac{\mathrm{d}\sigma^{(n)}(\boldsymbol{\xi}, \tilde{\boldsymbol{\xi}})}{\mathrm{d}y_+\,\mathrm{d}\varphi_+} = \frac{r_e^2}{x}\,\bar{F}_0^{(n)}\,. \tag{79}$$

Integrating this expression over φ_+ and then over y_+, we obtain

$$\begin{aligned} \frac{\mathrm{d}\sigma^{(n)}(\boldsymbol{\xi}, \tilde{\boldsymbol{\xi}})}{\mathrm{d}y_+} &= \frac{2\pi r_e^2}{x}\,\langle \bar{F}_0^{(n)} \rangle\,, \\ \sigma^{(n)}(\boldsymbol{\xi}, \tilde{\boldsymbol{\xi}}) &= \frac{2\pi r_e^2}{x}\int_{1-y_n}^{y_n} \langle \bar{F}_0^{(n)} \rangle\,\mathrm{d}y_+\,. \end{aligned} \tag{80}$$

To find $\langle \bar{F}_0^{(n)} \rangle$ we have to know the azimuthal dependence of the Stokes parameters ξ_i for the high-energy photon. These parameters are defined with respect to the polarization vectors $\mathbf{e}_\perp^{(1)}$ and $(-\mathbf{e}_\perp^{(2)})$ given by (26), i.e. they are defined with respect to the $x'y'z'$-axes which are fixed to the scattering plane. The x'-axis is along $\mathbf{e}_\perp^{(1)}$ and perpendicular to the scattering plane. The y'-axis is directed along $(-\mathbf{e}_\perp^{(2)})$ and, therefore, it is in that plane. Let $\check{\xi}_i$ be the Stokes parameters for the high-energy photons, fixed to the xyz-axes of the collider system. These Stokes parameters are connected with ξ_i by the following relations:

$$\xi_1 = -\check{\xi}_1 \cos 2\varphi_+ + \check{\xi}_3 \sin 2\varphi_+\,, \quad \xi_2 = \check{\xi}_2\,,$$
$$\xi_3 = -\check{\xi}_3 \cos 2\varphi_+ - \check{\xi}_1 \sin 2\varphi_+\,. \tag{81}$$

Inserting these relations into the effective cross section, we find

$$\langle \bar{F}_0^{(n)} \rangle \tag{82}$$
$$= (u-2)\, f_n + \frac{s_n^2}{1+\xi^2}\, g_n - (u-2)\, c_n\, h_n\, \xi_2\, P_c$$

for the case of the circularly polarized laser photons and

$$\langle \bar{F}_0^{(n)} \rangle = (u-2)\, \langle \tilde{f}_n \rangle + \frac{s_n^2}{1+\xi^2}\, \langle \tilde{g}_n \rangle$$
$$- \Big[2\langle \tilde{f}_n \rangle - (2 + 2c_n - s_n^2 \varDelta)\, \langle \tilde{g}_n \rangle$$
$$+ (1+c_n)^2\, \langle \tilde{g}_n \cos 2\varphi \rangle \Big]\, \check{\xi}_3 \tag{83}$$

for the case of the linearly polarized laser photons.[5] This means that for the circularly polarized laser photons the differential $\mathrm{d}\sigma^{(n)}(\boldsymbol{\xi},\, \tilde{\boldsymbol{\xi}})/\mathrm{d}y_+$ and the total $\sigma^{(n)}(\boldsymbol{\xi},\, \tilde{\boldsymbol{\xi}})$ cross sections for a given harmonic n do not depend on the degree of the linear polarization of the high-energy photon while for the linearly polarized laser photons these cross sections do not depend on the degree of the circular polarization of the high-energy photon.

The polarization vector of the final $e^\pm$ is determined by (47)– (49), but, unfortunately, the unit vectors $\mathbf{n}_j^\pm$ in this equation have no simple form. Therefore, we have to find the characteristics that are usually used for description of the positron (electron) polarization – the mean helicity $\langle \lambda_\pm \rangle$ and its transverse (to the vector $\mathbf{p}_\pm$) polarization $\boldsymbol{\zeta}_\perp^\pm$ in the considered collider system. For this purpose we introduce unit vectors $\boldsymbol{\nu}_j^\pm$, one of them is directed along the momentum of the final lepton $\mathbf{p}_\pm$ and two others are in the plane transverse to this direction:

$$\boldsymbol{\nu}_1^\pm = \frac{\mathbf{k}_1 \times \mathbf{p}_\pm}{|\mathbf{k}_1 \times \mathbf{p}_\pm|}\,, \quad \boldsymbol{\nu}_2^\pm = \frac{\mathbf{p}_\pm \times \boldsymbol{\nu}_1^\pm}{|\mathbf{p}_\pm \times \boldsymbol{\nu}_1^\pm|}, \quad \boldsymbol{\nu}_3^\pm = \frac{\mathbf{p}_\pm}{|\mathbf{p}_\pm|}\,;$$
$$\boldsymbol{\nu}_i^+\, \boldsymbol{\nu}_j^+ = \delta_{ij}\,, \quad \boldsymbol{\nu}_i^-\, \boldsymbol{\nu}_j^- = \delta_{ij}\,, \tag{84}$$

Therefore, $\boldsymbol{\zeta}_\pm\, \boldsymbol{\nu}_1^\pm$ is the transverse polarization of the final $e^\pm$ perpendicular to the scattering plane, $\boldsymbol{\zeta}_\pm\, \boldsymbol{\nu}_2^\pm$ is the transverse polarization in that plane and $\boldsymbol{\zeta}_\pm\, \boldsymbol{\nu}_3^\pm$ is the doubled mean helicity of the final electron:

$$\boldsymbol{\zeta}_\pm\, \boldsymbol{\nu}_3^\pm = 2\langle \lambda_\pm \rangle\,. \tag{85}$$

Let us discuss the relation between the scalar products $\boldsymbol{\zeta}_\pm \boldsymbol{\nu}_j^\pm$, defined above, and the invariants $\zeta_j^\pm$, defined in (33) and (35). In the collider system the vectors $\boldsymbol{\nu}_1^\pm$ and $\mathbf{n}_1^\pm$ coincide

$$\boldsymbol{\nu}_1^\pm = \mathbf{n}_1^\pm\,; \tag{86}$$

therefore,

$$\boldsymbol{\zeta}_\pm\, \boldsymbol{\nu}_1^\pm = \zeta_1^\pm\,. \tag{87}$$

Two other unit vectors $\mathbf{n}_2^\pm$ and $\mathbf{n}_3^\pm$ are in the scattering plane and they can be obtained from the vectors $\boldsymbol{\nu}_2^\pm$ and $\boldsymbol{\nu}_3^\pm$ by the rotation around the axis $\boldsymbol{\nu}_1^\pm$ on the angle $(-\varDelta\theta_\pm)$:

$$\boldsymbol{\zeta}_\pm\, \boldsymbol{\nu}_2^\pm = \zeta_2^\pm\, \cos\varDelta\theta_\pm + \zeta_3^\pm\, \sin\varDelta\theta_\pm\,,$$
$$\boldsymbol{\zeta}_\pm\, \boldsymbol{\nu}_3^\pm = \zeta_3^\pm\, \cos\varDelta\theta_\pm - \zeta_2^\pm\, \sin\varDelta\theta_\pm\,, \tag{88}$$

where

$$\cos\varDelta\theta_\pm = \boldsymbol{\nu}_3^\pm \mathbf{n}_3^\pm\,, \quad \sin\varDelta\theta_\pm = -\boldsymbol{\nu}_3^\pm \mathbf{n}_2^\pm\,. \tag{89}$$

As a result, the polarization vector of the final positron (electron) (38) is expressed as follows:

$$\boldsymbol{\zeta}_\pm^{(f)} = \boldsymbol{\nu}_1^\pm\, \frac{G_1^\pm}{\bar{F}_0} + \boldsymbol{\nu}_2^\pm \left(\frac{G_2^\pm}{\bar{F}_0} \cos\varDelta\theta_\pm + \frac{G_3^\pm}{\bar{F}_0} \sin\varDelta\theta_\pm \right)$$
$$+ \boldsymbol{\nu}_3^\pm \left(\frac{G_3^\pm}{\bar{F}_0} \cos\varDelta\theta_\pm - \frac{G_2^\pm}{\bar{F}_0} \sin\varDelta\theta_\pm \right)\,. \tag{90}$$

4.2 Approximate formulae

All the above formulae are exact. In this subsection we give some approximate formulae useful for application to the important case of high-energy $\gamma\gamma$ and γe colliders. It is expected (see, for example, the TESLA project [16]) that in the conversion region of these colliders, an electron with an energy of 250 GeV performs collisions with laser photons having the energy ≈ 1 eV per a single photon. The produced high-energy photons with the energy $\omega_1 \le 200$ GeV can further collide with the laser photon of energy $\omega_2 \approx 1$ eV, therefore, the invariant $x = 4\omega_1\omega_2/m^2 \le 4$ corresponds to the case when the e^+e^- pair can be produced only at $n \ge 2$. Therefore, the most interesting region of parameter x is around the value $x \approx 4$. The final electrons and positrons are ultra-relativistic and they are produced at a small angles $\theta_\mp$ with respect to the z-axis (chosen along the momentum $\mathbf{k}_1$ of the high-energy photon):

$$\omega_1 \gg m\,, \quad E_\pm \gg m, \quad \theta_\pm \ll 1\,. \tag{91}$$

In this approximation we have

$$y_\pm \approx \frac{E_\pm}{\omega_1}\,; \tag{92}$$

therefore, (79) gives us the distribution of the final positrons over their energy and the azimuthal angle. Besides, the lepton polar angle for the reaction (2) is

$$\theta_\pm \approx \frac{m\sqrt{nx}}{\omega_1}\; \frac{\sqrt{(y_n - y_\pm)(y_\pm - 1 + y_n)}}{y_\pm} \tag{93}$$

[5] Recall that in the case of linearly polarized laser photon we choose x-axis along the direction of laser linear polarization vector $\mathbf{e}_\mathrm{L}$.

and $\theta_\pm \to 0$ at $y_\pm \to y_n$ or at $y_\pm \to 1-y_n$. The maximal value of this angle is small

$$\max\{\theta_\pm\} \approx \frac{2n\,\omega_2}{m_*}\sqrt{1-\frac{4(1+\xi^2)}{nx}} \quad \text{at } y_\pm = \frac{2(1+\xi^2)}{nx}\,. \tag{94}$$

It is not difficult to check that, for the considered case, the angle $\Delta\theta_\pm$ between vector $\boldsymbol{\nu}_3^\pm$ and $\mathbf{n}_3^\pm$ is very small:

$$\Delta\theta_\pm \approx |\boldsymbol{\nu}_3^\pm \times \mathbf{n}_3^\pm| \approx \frac{m\theta_\pm}{2E_\pm} \le \frac{nm\omega_2}{m_* E_\pm}\sqrt{1-\frac{4(1+\xi^2)}{nx}}\,. \tag{95}$$

This means that the invariants $\zeta_j^\pm$ in (33) and (35) almost coincide with the projections $\boldsymbol{\zeta}_\pm$ on the vectors $\boldsymbol{\nu}_j^\pm$, defined in (84),

$$\zeta_1^\pm \approx -\zeta_\perp^\pm \sin(\varphi_\pm - \beta_\pm)\,, \quad \zeta_2^\pm \approx -\zeta_\perp^\pm \cos(\varphi_\pm - \beta_\pm)\,, \quad \zeta_3^\pm \approx 2\langle\lambda_\pm\rangle\,, \tag{96}$$

where $\beta_\pm$ is the azimuthal angle of the vector $\boldsymbol{\zeta}_\pm$ and $\zeta_\perp^\pm$ stands for the transverse components of the vectors $\boldsymbol{\zeta}_\pm$ with respect to the vector $\mathbf{p}_\pm$. Moreover, it means that the exact equation (90) for the polarization of the final positron and electron can be replaced with a high accuracy by the approximate equation

$$\boldsymbol{\zeta}_\pm^{(f)} \approx \sum_{j=1}^{3} \frac{G_j^\pm}{\bar{F}_0}\,\boldsymbol{\nu}_j^\pm\,. \tag{97}$$

Up to now we deal with the head-on collisions of the laser photon and the high-energy photon, when the collision angle α_0 between the vectors $\mathbf{k}_1$ and $(-\mathbf{k}_2)$ was equal zero. The detailed consideration of the case $\alpha_0 \ne 0$ is given in Appendix B of [15]. We present here the summary of such a consideration. If $\alpha_0 \ne 0$, the longitudinal component of the vector $\mathbf{k}_2$ becomes $(k_2)_z = -\omega_2\cos\alpha_0$ and it appeared to be the transverse (to the momentum $\mathbf{k}_1$) component $(\mathbf{k}_2)_\perp$. However this transverse component is small, $|(\mathbf{k}_2)_\perp| \lesssim \omega_2 \ll m$, therefore the transverse momenta of the final particles almost compensate each other, $(\mathbf{p}_-)_\perp = (\mathbf{k}_2 - \mathbf{p}_+)_\perp \approx -(\mathbf{p}_+)_\perp$. The invariants x and $y_\pm$ become (compare with (78) and (92))

$$x \approx \frac{4\omega_1\omega_2}{m^2}\cos^2\frac{\alpha_0}{2}\,, \quad y_\pm \approx \frac{E_\pm}{\omega_1}\,. \tag{98}$$

The polarization parameters of the initial photons and final electrons (positrons) conserve their forms (50), (65) and (81), (97). Therefore, the whole dependence on α_0 enters the effective cross section and the polarizations only via the quantity x (98).

5 Limiting cases

In this section we consider several limiting cases in which description of the non-linear Breit–Wheeler process is essentially simplified.

(i) The case of $\xi^2 \to 0$. At small ξ^2 all harmonics with $n > 1$ disappear due to the properties (53), (55) and (71), and we have

$$\mathrm{d}\sigma^{(n)}(\boldsymbol{\xi}, \tilde{\boldsymbol{\xi}}, \boldsymbol{\zeta}_\pm) \propto \xi^{2(n-1)} \quad \text{at} \quad \xi^2 \to 0\,. \tag{99}$$

The corresponding expression for $\mathrm{d}\sigma^{(1)}$ is given in the appendix.

(ii) The case of $y_\pm \to y_n$ for the circularly polarized laser photons. In this limit $y_\mp \to 1 - y_n$ and

$$\mathbf{p}_\perp \to 0\,, \quad s_n \to 0\,, \quad c_n \to -1\,. \tag{100}$$

In the collider system with $\omega_1 \gg \omega_2$ both leptons fly along the momentum of the high-energy photon $\mathbf{k}_1$ with $(q_\pm)_z \approx (p_\pm)_z > 0$. It means that

$$\mathbf{n}_3^+ = \mathbf{n}_3^- = \boldsymbol{\nu}_3^+ = \boldsymbol{\nu}_3^-\,, \quad \mathbf{n}_{1,2}^+ = -\mathbf{n}_{1,2}^- = \boldsymbol{\nu}_{1,2}^+\,,$$

therefore, $\zeta_3^\pm$ coincides with the doubled mean $e^\pm$ helicity,

$$\zeta_3^\pm = 2\langle\lambda_\pm\rangle\,, \quad \zeta_1^+\zeta_1^- + \zeta_2^+\zeta_2^- = -(\boldsymbol{\zeta}_+)_\perp(\boldsymbol{\zeta}_-)_\perp\,. \tag{101}$$

All harmonics with $n > 1$ disappear[6] and only the leptons of the first harmonic can be produced strictly along the direction of the high-energy photon momentum:

$$\frac{\mathrm{d}\sigma^{(n)}(\boldsymbol{\xi}, \tilde{\boldsymbol{\xi}}, \boldsymbol{\zeta}_\pm)}{\mathrm{d}y_+\mathrm{d}\varphi_+} = \frac{r_e^2}{4x}\,\bar{F}^{(n)} \propto f_n \propto (y_n - y_+)^{n-1} \quad \text{at} \quad y_\pm \to y_n\,, \tag{102}$$

with

$$\frac{\bar{F}^{(n)}}{f_n} = (1+\xi_2 P_c) \times\left[\frac{y_n^2 + (1-y_n)^2}{y_n(1-y_n)}\,(1-\zeta_3^+\zeta_3^-) - 2(\boldsymbol{\zeta}_+)_\perp(\boldsymbol{\zeta}_-)_\perp\right] \pm \frac{2y_n - 1}{y_n(1-y_n)}\,(\xi_2 + P_c)\,(\zeta_3^+ - \zeta_3^-) \tag{103}$$

(the $\pm$ sign in the last term of (103) corresponds to $y_\pm \to y_n$). In this limit, the final leptons are longitudinal polarized,

$$\zeta_3^{(n)(f)+} = -\zeta_3^{(n)(f)-} = \pm\frac{2y_n - 1}{y_n^2 + (1-y_n)^2}\,\frac{\xi_2 + P_c}{1 + \xi_2 P_c}\,,$$

[6] The vanishing of all harmonics with $n \ge 3$ is due to the conservation of z component of the total angular momentum J_z in this limit: the initial value of $J_z = \lambda_1 - n\,\lambda_2$ can not be equal for $n \ge 3$ to the final value of $J_z = \lambda_+ + \lambda_-$ (here λ_1 (λ_2) is the helicity of the high-energy (laser) photon and $\lambda_\pm$ is the helicity of the positron (electron)). For $n = 2$ this argument does not help, since $J_z = \pm 1$ can be realized for the initial and final states. The vanishing of the second harmonics for $y_\pm \to y_n$ is a specific feature of the process, related to the facts that the polarization vectors $e_1^{(\lambda_1)}$ and $e_2^{(\lambda_2)}$ are orthogonal not only to the 4-vectors k_1, k_2 but to the 4-vectors $p_\pm$ as well and that $e_1^{(\lambda_1)}\,e_1^{(\lambda_1)} = 0$.

$$\boldsymbol{\zeta}_\perp^{(n)(f)\pm} = 0\,. \qquad (104)$$

(iii) The case of large $x \gg 1+\xi^2$ for the circularly polarized laser photons. In this case $1-y_n \approx (1+\xi^2)/(nx) \ll 1$. The spectrum of the fist harmonic is symmetric under replacement $y_+ \leftrightarrow 1-y_+$ and has peaks at maximum or minimum value of y_+. The form of these peaks at not too small transverse momentum of leptons, at $m_*^2 \ll \mathbf{p}_\perp^2 \ll m^2x$, is

$$\frac{\mathrm{d}\sigma^{(1)}(\boldsymbol{\xi},\tilde{\boldsymbol{\xi}})}{\mathrm{d}y_+} = \frac{2\pi r_e^2}{x}(1-\xi_2P_c)\left[\frac{1}{y_+(1-y_+)}-2\right]. \qquad (105)$$

The magnitude of these peaks increases with growth of x. Therefore, the regions near this peaks give the dominant contribution to the cross section for the first harmonic. On the other hand, all other harmonics vanish in these regions. As a result, the cross section for the first harmonic $\sigma^{(1)}$ approximates well the total cross section σ summed over all harmonics. Thus, we consider the regions

$$m_*^2 \ll \mathbf{p}_\perp^2 \ll m^2x\,, \qquad (106)$$

in which $y_\pm$ is close to y_1 or to $1-y_1$ while y_1 is close to unit:

$$1-y_1 \approx \frac{1+\xi^2}{x} \ll 1\,,\ \ |s_n| \ll 1\,,\ \ c_n \approx +1\,. \qquad (107)$$

In these regions the function $\bar F^{(1)}$ becomes large

$$\bar F^{(1)} \approx \frac{1}{(1-y_+)(1-y_-)} \times \left[(1-\xi_2P_c)(1-\zeta_3^+\zeta_3^-) \pm (\xi_2-P_c)\,(\zeta_3^+-\zeta_3^-)\right] \qquad (108)$$

(for $y_\pm$ near y_1). Therefore, integrating the effective cross section for the first harmonic, $\mathrm{d}\sigma^{(1)} \propto 1/[(1-y_+)(1-y_-)]$, near its two maximums, i.e. in the regions

$$\frac{1+\xi^2}{x} \ll 1-y_\pm \ll 1\,, \qquad (109)$$

we find (with logarithmic accuracy) the total cross section

$$\sigma(\boldsymbol{\xi},\tilde{\boldsymbol{\xi}},\boldsymbol{\zeta}_\pm) = \frac{\pi r_e^2}{x}\,(1-\xi_2P_c)(1-\zeta_3^+\zeta_3^-)\,\ln\frac{x}{1+\xi^2}\,. \qquad (110)$$

The total cross section, summed over spin states of the final particles, has the form

$$\sigma(\boldsymbol{\xi},\tilde{\boldsymbol{\xi}}) = \frac{4\pi r_e^2}{x}\,(1-\xi_2P_c)\,\ln\frac{x}{1+\xi^2}\,. \qquad (111)$$

(iv) The case of $y_\pm \to y_n$ for the linearly polarized laser photons. In this limit, the argument a of the functions (68) tends to zero, while the argument b tends to the non-zero constant, $b \to n\xi^2/[2(1+\xi^2)]$; the function

$$\tilde f_n = \left[J_{(n-1)/2}(b) - J_{(n+1)/2}(b)\right]^2 \qquad (112)$$

for odd n and

$$\tilde f_n = \frac{2}{\xi^2}\left[J_{n/2}(b)\right]^2 \qquad (113)$$

for even n. The spectra and the polarizations are determined by the function $\bar F^{(n)}$ given for odd harmonics by the expression

$$\frac{\bar F^{(n)}}{\tilde f_n} = (u_n-2-2\check\xi_3)(1-\zeta_3^+\zeta_3^-) - \left[2-(u_n-2)\check\xi_3\right](\boldsymbol{\zeta}_+)_\perp(\boldsymbol{\zeta}_-)_\perp \pm (2y_n-1)u_n\left[\xi_2\,(\zeta_3^+-\zeta_3^-) + \check\xi_1(\zeta_1^+\zeta_2^- - \zeta_2^+\zeta_1^-)\right] \qquad (114)$$

and for even harmonics by the expression

$$\bar F^{(n)} = u_n\,\tilde f_n[1+\zeta_3^+\zeta_3^- + \xi_3(\zeta_1^+\zeta_1^- - \zeta_2^+\zeta_2^-) + \xi_2\,(\zeta_3^+ + \zeta_3^-) + \xi_1(\zeta_1^+\zeta_2^- + \zeta_2^+\zeta_1^-)] \qquad (115)$$

with $u_n = 1/[y_n(1-y_n)]$.

(v) The behavior of the cross section near the threshold

$$x \to x_n = \frac{4(1+\xi^2)}{n}\,. \qquad (116)$$

In this limit

$$y \to 1/2\,,\ \ \mathbf{p}_\perp \to 0\,,\ \ s_n \to 0\,,\ \ c_n \to -1 \qquad (117)$$

and the main results can be obtain from the case $y_+ \to y_n$ just by substitution $y_n = 1/2$ into (102)– (104) and (112)–(115). In particular, for the first harmonic with circularly polarized laser photons we have

$$\bar F^{(1)} = 2(1+\xi_2P_c)\left(1-\boldsymbol{\zeta}_+\boldsymbol{\zeta}_-\right). \qquad (118)$$

It is interesting to compare the behavior of the first harmonic near the threshold and far from the threshold. Let us consider the pure quantum states for the photons (their helicities $|\lambda_{1,2}| = 1$) and $e^\pm$ (their helicities $|\lambda_\pm| = 1/2$) in the collider system with $\omega_1 \gg \omega_2$. It follows from (118) that near the threshold the photons have the same helicities while the leptons have opposite helicities:

$$\lambda_1 = \lambda_2\,,\ \ \lambda_+ = -\lambda_- \ \text{ at } \ x \to x_1 = 4(1+\xi^2)\,. \qquad (119)$$

If $x \gg x_1$, we have to distinguish two regions. In the region (100) of very small transverse momenta of the leptons $\mathbf{p}_\perp \to 0$, we have the same relations (119) [see (103)], but in the region (106) of relatively large transverse momenta of the leptons the photons have opposite helicities as well as the leptons [see (108)]:

$$\lambda_1 = -\lambda_2\,,\ \ \lambda_+ = -\lambda_- \ \ \text{at } \ x \gg x_1\,,\ \ m^2x_1 \ll \mathbf{p}_\perp^2 \ll m^2x\,. \qquad (120)$$

6 Numerical results related to $\gamma\gamma$ and γe colliders

In this section we present some examples which illustrate the formulae obtained in the previous sections. In the figures below we use the notation

$$\sigma_0 = \pi r_e^2 \approx 2.5\cdot 10^{-25}\,\mathrm{cm}^2\,.$$

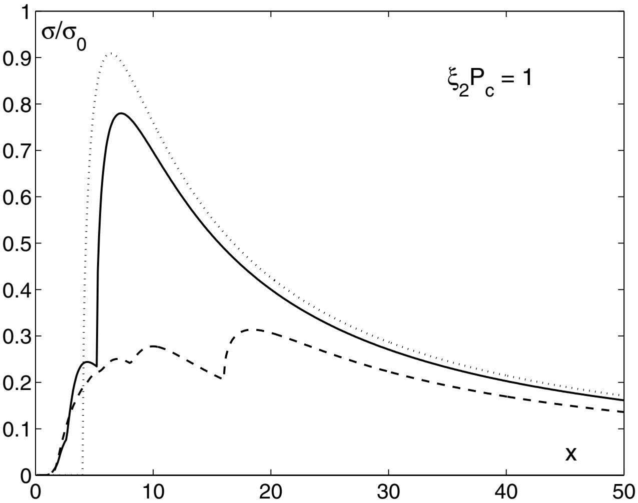

Fig. 1. The total effective cross section of the non-linear Breit–Wheeler process in dependence on $x = 4\omega_1\omega_2/m^2$. The laser and high-energy photons have the same helicity, $P_c = \xi_2 = 1$. The dotted curve corresponds to $\xi^2 = 0$, the solid curve to $\xi^2 = 0.3$ and the dashed curve to $\xi^2 = 3$

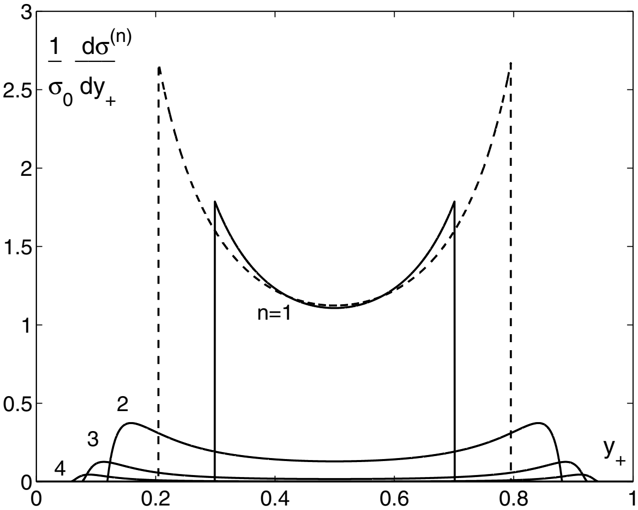

Fig. 3. Energy spectra of positrons for different harmonics n at $x = 6.2$ and $\xi^2 = 0.3$. The laser and high-energy photons have the same helicities, $P_c = \xi_2 = 1$. The dashed curves correspond to $\xi^2 = 0$

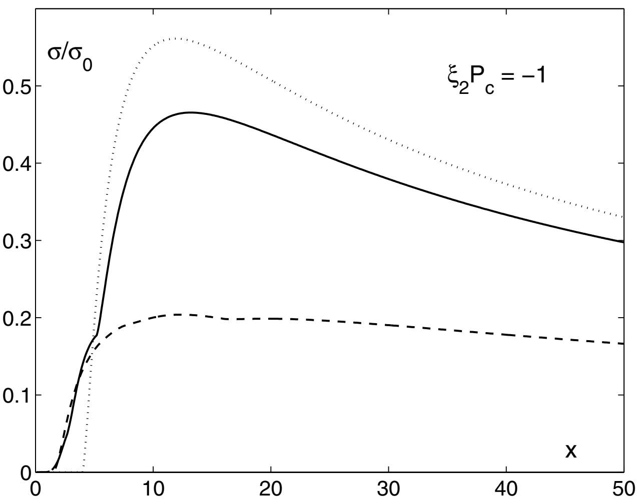

Fig. 2. The same as in Fig. 1, but for $P_c = -\xi_2 = 1$

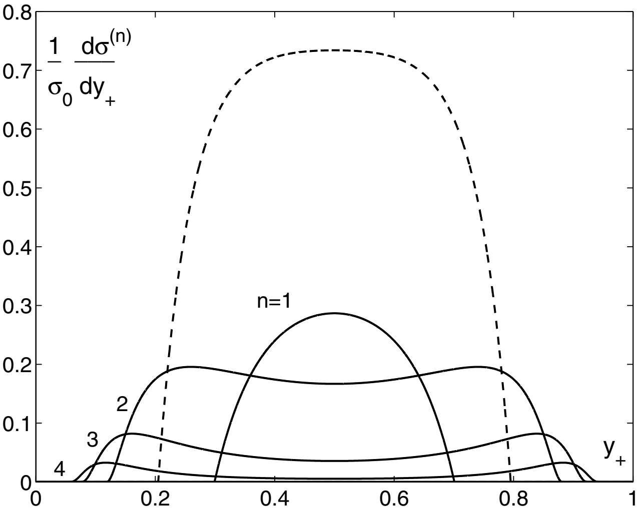

Fig. 4. The same as in Fig. 3, but for $P_c = -\xi_2 = 1$

The total cross section for the most interesting case of the circularly polarized laser photons is shown in Figs. 1 and 2 in dependence on the parameter $x = 4\omega_1\omega_2/m^2$ for three values of the non-linearity parameter ξ^2 (defined in (6)): for $\xi^2 = 0$ (the linear Breit–Wheeler process (1)), for $\xi^2 = 0.3$ (as in the TESLA project) and larger by one order of magnitude, $\xi^2 = 3$. The linear Breit–Wheeler process has a clear threshold: a pair production is possible only at $x > x_1 = 4$. At non-zero value of ξ^2, the electron pair production becomes possible for any $x > 0$ due to excitation of the higher harmonics. Note that with the increase of ξ^2 the total effective cross section decreases.

It is seen from (80) and (82) that all these cross sections depend on the polarization of the initial photons only via the factor $\xi_2 P_c$. When the helicities of the initial photons are the same ($\xi_2 P_c = 1$; see Fig. 1), these cross sections dominate in the region of not too large x (approximately at $x < 15$), at larger x the cross sections with opposite helicities of the initial photons ($\xi_2 P_c = -1$; see Fig. 2) become dominant.

The following figures illustrate the dependence of the differential cross sections $\mathrm{d}\sigma^{(n)}(\boldsymbol{\xi}, \tilde{\boldsymbol{\xi}})/\mathrm{d}y_+$ and the positron polarizations $\zeta_j^{(n)(f)+}$ on the relative positron energy $y_+ \approx E_+/\omega_1$. The non-linearity parameter is chosen to be the same as in the TESLA project, $\xi^2 = 0.3$; therefore, the threshold value of x for the first harmonic is $x_1 = 4(1+\xi^2) = 5.2$. We chose the value of parameter $x = 6.2$, which is a one unit above the threshold x_1.

Let us now turn to the case of the *circularly* polarized laser photons (Figs. 3, 4, 5 and 6).

The spectra of the few first harmonics are shown in Figs. 3 and 4. When helicities of the initial photons are the same ($\xi_2 P_c = 1$; see Fig. 3), the main contribution (for the chosen $\xi^2 = 0.3$) is given by the first harmonic, the probabilities for generation of the higher harmonics are

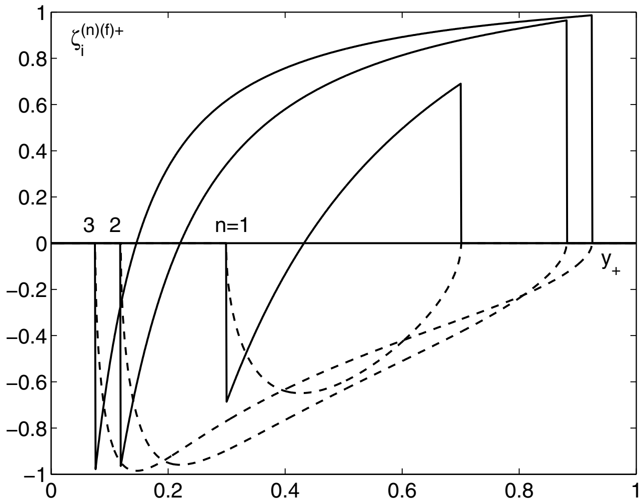

Fig. 5. Polarization of positrons for different harmonics n at $x = 6.2$ and $\xi^2 = 0.3$. The laser and high-energy photons have the same helicity, $P_c = \xi_2 = 1$. The solid curves correspond to $\zeta_3^{(n)(f)+} \approx 2\langle\lambda_+\rangle$, the dashed curves correspond to $\zeta_2^{(n)(f)+}$ which is the transverse electron polarization in the scattering plane; the transverse electron polarization perpendicular to the scattering plane $\zeta_1^{(n)(f)+} = 0$ in this case

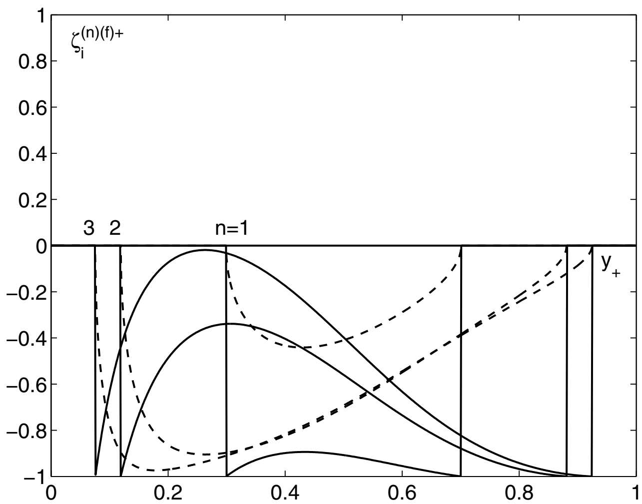

Fig. 6. The same as in Fig. 5, but for $P_c = -\xi_2 = 1$

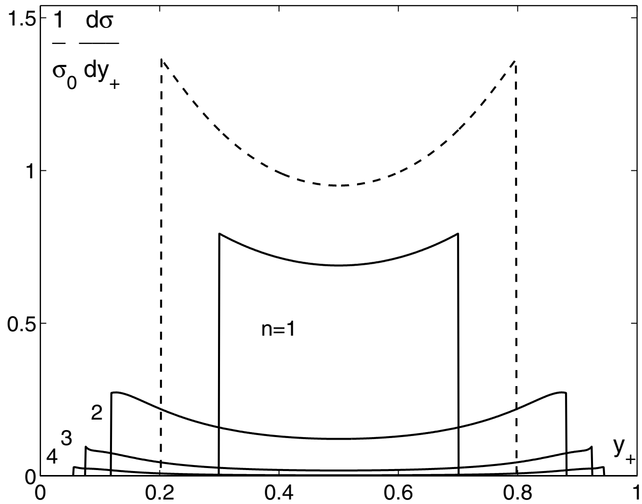

Fig. 7. Energy spectra of positrons for different harmonics n at $x = 6.2$ and $\xi^2 = 0.3$. The laser photons are linearly polarized, the high-energy photon is circularly polarized $\xi_2 = 1$. The dashed curves correspond to $\xi^2 = 0$

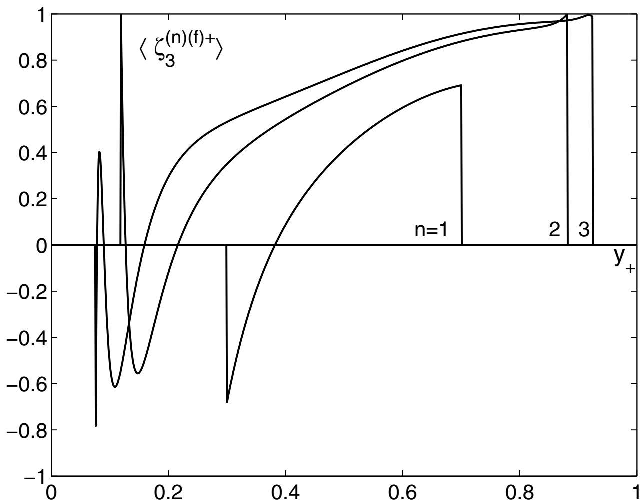

Fig. 8. The longitudinal polarization $\langle\zeta_3^{(n)(f)+}\rangle = \langle G_3^{(n)+}\rangle / \langle \bar{F}_0^{(n)}\rangle$ of positrons for different harmonics n at $x = 6.2$ and $\xi^2 = 0.3$, averaged over the azimuthal angle φ_+. The laser photons are linearly polarized, the high-energy photon is circularly polarized $\xi_2 = 1$. The averaged transverse electron polarizations $\langle \boldsymbol{\zeta}_\perp^{(n)(f)+}\rangle = 0$ in this case

smaller. When helicities of the initial photons are opposite ($\xi_2 P_c = -1$; see Fig. 4), the first and the second harmonics give approximately equal contributions. The changes in the spectra for the transition from $\xi^2 = 0$ to $\xi^2 = 0.3$ are more pronounced for the case $\xi_2 P_c = -1$ than for $\xi_2 P_c = 1$.

The polarization of positrons for these two cases are shown in Figs. 5 and 6, respectively. It is seen that in the high-energy part of the spectrum the longitudinal polarization of positrons $\zeta_3^{(n)(f)+}$ is large for all harmonics and that its sign coincides with the sign of the high-energy photon helicity ξ_2.

Next we will discuss the case of the *linearly* polarized laser photons (Figs. 7 and 8).

The spectra of the first few harmonics for this case are shown in Fig. 7. They differ considerably from those for the case of the circularly polarized laser photons shown in Fig. 3. First of all, in the considered case the spectra depend on the polarization of the high-energy photon only via the factor $\tilde{\xi}_3$ defined in (81). The maximum of the first harmonic at $y = y_1$ now is about two times smaller than that on Fig. 3. Besides, the harmonics with $n > 1$ do not vanish at $y = y_n$ contrary to such harmonics on Fig. 3.

The polarization of positrons is not equal zero only if the high-energy photon is circularly polarized. In this case the longitudinal polarization of positrons is large in the high-energy part of their spectrum (see Fig. 8), while the transverse positron polarization, averaged over positrons azimuthal angle φ_+, equals zero.

7 Summary and comparison with other papers

Our main results are given by (56)– (60) for the circularly and by (74)– (77) for the linearly polarized laser beam. They are expressed in terms of the 16 functions $\bar{F}_0$, $G_j^{\pm}$ and $\bar{H}_{ij}$ with $i,\, j = 1 \div 3$, which describe completely the polarization properties of the non-linear Breit–Wheeler process in a compact invariant form. The function $\bar{F}_0$ enters the total cross section (46) and (80) and the differential cross sections (79) and (80). The polarization of the final positrons (electrons) is described by functions $G_j^{\pm}$, which enter the polarization vector $\boldsymbol{\zeta}_{\pm}^{(f)}$ given by the exact equations (48) and (90) and the approximate equations (97).

We considered the kinematics and the approximate formulae relevant for the problem of $e \to \gamma$ conversion at the $\gamma\gamma$ and γe colliders. Besides, we discuss the spectra and the polarization of the final particles in the limit of small and large energies (Sect. 5). For the circularly polarized laser photons and for large values of the parameter $x \gg 1 + \xi^2$, we obtain (with logarithmic accuracy) the simple analytical expression for the total cross section, (110) and (111). In Sect. 6 we present the numerical results and discuss several interesting features, which may be useful for the analysis of the background and the luminosity of γe and $\gamma\gamma$ colliders.

Let us compare our results with those obtained earlier.

Circular polarization of laser photons. In the literature we found the results which can be compared with our functions $\bar{F}_0$ and $G_3^{\pm}$, namely, in [9, 10] (the function $\bar{F}_0$), in [11–13] (the functions $\bar{F}_0$ and $G_3^{\pm}$ only at $\zeta_1^{\pm} = \zeta_2^{\pm} = \xi_1 = \xi_3 = 0$).

The function $\bar{F}_0^{(n)}$ (56) can be presented in the form

$$\bar{F}_0^{(n)} = \langle \bar{F}_0^{(n)} \rangle - C^{(n)} \xi_3 \,, \tag{121}$$

where $\langle \bar{F}_0^{(n)} \rangle$ is given by (82) or (using (81)) in the form

$$\begin{aligned} \bar{F}_0^{(n)} &= \langle \bar{F}_0^{(n)} \rangle \\ &\quad + C^{(n)} \check{\xi}_3 \cos 2\varphi_+ + C^{(n)} \check{\xi}_1 \sin 2\varphi_+ \,. \end{aligned} \tag{122}$$

Our function $\langle \bar{F}_0^{(n)} \rangle$ coincide with those obtained in [8, 10, 12, 13], but our expression for $C^{(n)}$,

$$\begin{aligned} C^{(n)} &= 2(f_n - g_n) + s_n^2 (1 + \varDelta)\, g_n \\ &= \frac{4}{\xi^2} J_n^2(z_n) + 4 \left[-\left(\frac{n^2}{z_n^2} - 1 \right) J_n^2(z_n) + \left(J_n'(z_n) \right)^2 \right] \end{aligned} \tag{123}$$

does not coincide with that given by (50) and (51) on p. 66 in [10]

$$\begin{aligned} &C_{\mathrm{R}}^{(n)} \\ &= \frac{4}{\xi^2} J_n^2(z_n) + 4 \left[+ \left(\frac{n^2}{z_n^2} - 1 \right) J_n^2(z_n) + \left(J_n'(z_n) \right)^2 \right] . \end{aligned} \tag{124}$$

However, our expression has a proper limit: $C^{(1)} = s_1^2$ at $\xi^2 \to 0$, while the expression (124) has a wrong limit $C_{\mathrm{R}}^{(1)} = 2 + s_1^2$ at $\xi^2 \to 0$. The difference is due to misprint in the paper [10]: the second term in (124) should have the sign “minus”.

For polarization of the final electron, our results differ slightly from those in [12]. Namely, function G_3^- in this paper coincide with our one. However, the polarization vector $\boldsymbol{\zeta}_{-}^{(f)}$ in the collider system is obtained in paper [12] not in the exact form (90), but only in the approximate form equivalent to our approximate equation (97).

Linear polarization of laser photons. In the literature we found the results which can be compared with our function $\bar{F}_0$ only. Our expression for $\bar{F}_0^{(n)}$ (74) is in agreement with that obtained in [8, 10].

The correlation of the final particles' polarizations are described by functions $\bar{H}_{ij}$ given in (60) for the circular, and in (77) for the linear laser polarization. To the best of our knowledge there is not any of the results in the literature to compare with (60) and (77).

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Appendix: Limit of the weak laser field

Let us consider the limit of $\xi^2 \to 0$, i.e. the process (1). The cross section of this process can be obtain from the cross section (42) at $\xi^2 \to 0$ and has the form

$$\begin{aligned} \mathrm{d}\sigma(\boldsymbol{\xi},\, \tilde{\boldsymbol{\xi}},\, \boldsymbol{\zeta}_{\pm}) &= \frac{r_e^2}{4x}\, \bar{F}\, \mathrm{d}\varGamma \,, \\ \mathrm{d}\varGamma &= \delta(k_1 + k_2 - p_+ - p_-)\, \frac{\mathrm{d}^3 p_+}{E_+} \frac{\mathrm{d}^3 p_-}{E_-} \,, \end{aligned} \tag{125}$$

where

$$\bar{F} = \bar{F}_0 + \sum_{j=1}^{3} \left(G_j^+ \zeta_j^+ \;+\; G_j^- \zeta_j^- \right) + \sum_{i,j=1}^{3} \bar{H}_{ij}\, \zeta_i^+ \, \zeta_j^- \,. \tag{126}$$

To obtain $\bar{F}$ from $\bar{F}^{(1)}$ (45), we should take into account that the Stokes parameters of the laser photon have the values (50) for the circular polarization and the values (65) for the linear polarization. Besides, our invariants c_1, s_1, r_1 and the auxiliary functions (73) transform at $\xi^2 \to 0$ to

$$\begin{aligned} & c_1 \to c = 1 - 2r \,, \quad s_1 \to s = 2\sqrt{r(1-r)} \,, \\ & r_1 \to r = \frac{u}{x} \,, \quad u = \frac{1}{y_+ y_-} \,, \\ & X_1 \to -c \,, \quad Y_1 \to c(1 - \tilde{\xi}_3) \,, \quad V_1 \to -\tilde{\xi}_3 \,. \end{aligned} \tag{127}$$

As a result, we obtain

$$\begin{aligned} \bar{F}_0 &= (u-2)(1 - c\, \xi_2 \tilde{\xi}_2) \, + s^2 (1 - \xi_3)(1 - \tilde{\xi}_3) \\ &\quad -2(c \xi_1 \tilde{\xi}_1 + \xi_3 \tilde{\xi}_3) \,; \end{aligned} \tag{128}$$

$$
\begin{aligned}
G_1^+ &= -\frac{s}{y_-}\xi_1\,\tilde\xi_2 + \frac{s}{y_+}\xi_2\,\tilde\xi_1\,,\\
G_2^+ &= \frac{s}{y_+}\left[c(1-\tilde\xi_3)\xi_2 - \tilde\xi_2\right] + \frac{s}{y_-}\,\xi_3\tilde\xi_2\,, \qquad (129)\\
G_3^+ &= (y_+ - y_-)u\,(\xi_2 - c\,\tilde\xi_2) + \frac{s^2}{y_+}\,\xi_2(1-\tilde\xi_3)\,;
\end{aligned}
$$

$$
\begin{aligned}
\bar H_{11} &= 2(1 - c\xi_2\tilde\xi_2) - (u-2)(c\xi_1\tilde\xi_1 + \xi_3\tilde\xi_3)\\
&\quad -s^2(1-\xi_3)(1-\tilde\xi_3)\,,\\
\bar H_{22} &= 2(1 - c\xi_2\tilde\xi_2) - (u-2)(c\xi_1\tilde\xi_1 + \xi_3\tilde\xi_3)\\
&\quad -s^2(1-\tilde\xi_3)[1 + (u-1)\xi_3]\,,\\
\bar H_{33} &= -(u-2)(1 - c\xi_2\tilde\xi_2) + 2(c\xi_1\tilde\xi_1 + \xi_3\tilde\xi_3)\\
&\quad +s^2(1-\tilde\xi_3)(u - 1 + \xi_3)\,, \qquad (130)\\
\bar H_{12} &= \frac{s^2}{y_-}\,\xi_1(1-\tilde\xi_3) + u(y_+ - y_-)(\xi_1\tilde\xi_3 - c\xi_3\tilde\xi_1)\,,\\
\bar H_{13} &= \frac{s}{y_+}\,\tilde\xi_1 - \frac{s}{y_-}\,[\xi_3\tilde\xi_1 + c\xi_1(1-\tilde\xi_3)]\,,\\
\bar H_{23} &= \frac{s}{y_-}[c\xi_3(1-\tilde\xi_3) - \xi_1\tilde\xi_1]\\
&\quad + \frac{s}{y_+}[c(1-\tilde\xi_3) - \xi_2\tilde\xi_2]\,.
\end{aligned}
$$

The rest of the functions, G_j^-, $\bar H_{21}$, $\bar H_{31}$ and $\bar H_{32}$, can be obtained from the above equations using the symmetry relations (62):

$$
\begin{aligned}
G_j^-(y_+, y_-) &= G_j^+(y_-, y_+)\,,\\
\bar H_{ij}(y_+, y_-) &= \bar H_{ji}(y_-, y_+)\,. \qquad (131)
\end{aligned}
$$

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